

Universality classes in the random-storage sandpile model

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The avalanche statistics in a stochastic sandpile model where toppling takes place with a probability p is investigated. The limiting case $p=1$ corresponds to the Bak-Tang-Wiesenfeld (BTW) model with a deterministic toppling rule. Based on the moment analysis of the distribution of avalanche sizes we conclude that for $0 < p < p_c$ the model belongs to the direct percolation universality class while for $p_c < p < 1$ it belongs to the BTW universality class, where p_c is identified with the critical probability for directed percolation in the corresponding lattice.

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Sandpile automata were proposed as a paradigm of self-organized critical (SOC) phenomena [1]. These simple models capture its essential dynamics, which takes place in the form of avalanches of all sizes. At the early state of SOC theory it was believed that the critical state of sandpile automata is insensitive to changes in model parameters; however, some recent works contradict this statement. For instance, Vespignani and Zapperi [2] have shown that driving and dissipation rates actually act as control parameters, criticality is obtained after fine tuning of these fields. On the other hand, we have recently shown that a class of models with stochastic rules display a transition from SOC to directed percolation (DP) with increasing the degree of stochasticity [3]. Nevertheless, before we make our final conclusion, we have to investigate if the original Bak-Tang-Wiesenfeld (BTW) sandpile automaton and these modified models belong to the same universality class; otherwise, they would just be different models. One may thus ask, do deterministic and stochastic sandpile models belong to the same universality class?

The BTW automaton is defined on a d -dimensional hypercubic lattice of linear size L . On each site i an integer variable z_i is defined, which we call energy following [2]. Energy is added to the system by selecting a site at random and increasing its energy by one, i.e., $z_i \rightarrow z_i + 1$. The energy addition continues until one of the sites reaches or exceeds a threshold $z_c = 2d$; then this site topples, transferring energy to its neighbors. Toppling is defined by the set of rules $z_i \rightarrow z_i - z_c$ and $z_j \rightarrow z_j + 1$ at each of the j nearest neighbors. The toppling at one site may induce an avalanche of toppling events. After the avalanche has stopped we restart adding energy to the system. Open boundary conditions are assumed. The toppling rules of this model are deterministic and the only randomness is introduced through the random energy addition.

Maybe the simplest stochastic sandpile model is the Manna or d -state model [4]. In this case $z_c = d < 2d$ and, therefore, only d of the $2d$ neighbors will receive energy after toppling. These d neighbors are selected at random. The real-space renormalization-group approach [5] suggests that the BTW and Manna models belong to the same universality class. However, there is not complete agreement in numerical simulations. Early large-scale simulations of the Manna

[4] and BTW models [6] in two dimensions show that the avalanche distributions are described by the same exponents for the power-law decay and the scaling of the cutoffs. These results were contradicted by more recent estimates of the avalanche exponents by Lübeck and Usadel [7]. On the other hand, Ben Hur and Biham [8] analyzed the scaling of conditional expectation values of various quantities, obtaining significant differences in the exponents for the two models. However, Chessa *et al.* have recently shown [9] that the method of conditional expectation values, introduced by Ben-Hur and Biham, is systematically biased by nonuniversal corrections and, therefore, does not provide indications on universality classes. Moreover, according to the large-scale simulations performed by Chessa *et al.* [9], the BTW and Manna models belong to the same universality class.

In the Manna model the randomness appears in the energy transfer after toppling, but the condition for toppling is deterministic. Motivated on a directed model by Tadic and Dhar [10] we have proposed a different model, where sites topple when $z \geq z_c = 2d$ but with probability p [3]. In this case, as in the BTW model, each of the $2d$ neighbors receives one energy grain but the condition for toppling is stochastic. In the particular case $p=1$ one recovers the BTW model. For $p < 1$ sites accumulate a random amount of energy, random-storage model (RSM) [11]. From mean-field theory and numerical simulations in one dimension [3] we have shown that the RSM self-organizes into a critical state for $p_c < p < 1$, while it is subcritical for $p < p_c$. The subcritical state is found to be similar to DP and p_c is identified with the critical threshold for DP. The correlation length exponents are identical to those of DP but other exponents results are different due to open boundary conditions. However, in that occasion our analysis was limited to a small region close to p_c and simulations were performed only in $d=1$. Thus, we could not provide any indication about the universality classes.

In the present paper we extend the numerical simulations of the RSM to the whole range of p , in one and two dimensions. To determine the avalanche size and duration exponents we use moment analysis, a technique previously introduced by de Menech *et al.* [12] to obtain the scaling exponents for the BTW model. techniques. The numerical evidence suggests that the RSM and the BTW model belong

to different universality classes. Moreover, for $p < p_c$ we corroborate that the model is similar to DP.

The meanfield (MF) theory of the RSM [3] reveals that there is a critical probability p_c below which the system is subcritical. In this regime p is small and sites accumulate a large amount of grains. Let us assume that all sites have heights above the critical threshold z_c . Then when a new grain is added at a certain site this site will topple with probability p . If it topples, then each of its nearest neighbors will also topple with probability p and so on. This picture is equivalent to a site-directed percolation problem, where sites become active with probability p if one of its neighbors was active in the previous step. Hence, the correlation length is given by $\xi \sim (p_c - p)^{-\nu}$, where p_c is the critical threshold for site-directed percolation in the corresponding lattice and ν is the spatial correlation length exponent. Now, if $p \ll p_c$ in such a way that $\xi \ll L$, where L is the system size, then an avalanche starting from a bulk site will never reach the boundary and, therefore, no grain will be dissipated. Hence, the average height will increase in time because dissipation will not balance the grain addition from the external field. After a time long enough most sites will have heights above z_c , supporting our starting assumption. The dynamical state below p_c is thus equivalent to DP and p_c is identified with the site DP threshold in the corresponding lattice. Above p_c and for $z > z_c$ at all sites the system will still be equivalent (at least for small times) to DP and an infinite avalanche will be generated. However, such a state is not stable because the infinite avalanche will reach the boundary leading to dissipation of grains and hence the decrease of the column heights up to the avalanche has stopped. Above p_c the stationary state is no more equivalent to DP. The balance between the grains added by the external field and dissipation at the boundary leads to a SOC state. Our task will be to determine if this SOC state belongs to the same universality class of the BTW model.

Let s be the avalanche size and T its duration. s is the number of toppling events in an avalanche. T is the number of steps required to obtain a stable configuration starting from the initial site with $z \geq z_c$, which triggered the avalanche, taking into account that on each step all sites are updated in parallel following the toppling rule. Both s and T are random variables. Their distributions are given by, assuming scaling,

$$P_x(x) = x^{-\tau_x} f_x(x/x_c), \quad (1)$$

where x is s or T , s_c and T_c are the cutoff avalanche size and duration, and f_x is a cutoff function. The cutoffs scale with the characteristic length of the system ξ according to

$$x_c \sim \xi^{\beta_x}, \quad (2)$$

where $\beta_s = D$, $\beta_t = z$, D is the avalanche dimension, and z is the dynamic scaling exponent. In the SOC state $\xi \sim L$ and, therefore, the cutoffs scale with system size. On the contrary, in the subcritical state $\xi \sim (p_c - p)^{-\nu}$ and, hence, the cutoffs diverges when p approaches p_c . The scaling exponents τ_x and β_x are not all independent. From the identity $P(s)ds = P(T)dt$ one obtains

$$(\tau_s - 1)D = (\tau_t - 1)z. \quad (3)$$

On the other hand, in the SOC state ($p > p_c$) the mean avalanche size should scale as $\langle s \rangle \sim L^2$ and, therefore,

$$(2 - \tau_s)D = 2. \quad (4)$$

These scaling relations are useful to test the reliability of the numerical estimates.

The purpose of present numerical simulations is to determine the exponents τ_s , τ_t , D , and z . We are going to take up this task using as fundamental technique the moment analysis [12]. The q moment is given by

$$\langle x^q \rangle = \int dx P(x) x^q \sim \xi^{\sigma_x(q)}, \quad (5)$$

where

$$\sigma_x(q) = \beta_x(1 - \tau_x) + \beta_x q. \quad (6)$$

The last equivalence in Eq. (5) is not valid for small values of q . For small q the integral depends on the functional form of $P(x)$ in the whole range of x , while scaling assumptions are, in general, not valid for small x . The extreme case is $q = 0$, normalization imposes $\sigma_x(0) = 0$ and, therefore, Eq. (6) is not valid. But, for $q \geq \tau_x - 1$ large x dominates leading to the last equivalence in Eq. (5).

After computing the moments one can obtain $\sigma_x(q)$ from a linear fit to the log-log plot of $\langle x^q \rangle$ vs ξ . Then one can obtain τ_x and β_x from a linear fit to the straight part of the plot $\sigma_x(q)$ vs q . Above p_c we have $\xi \sim L$, but it is a function of p below. To compute the correlation length below p_c we use the following expression [3]:

$$\xi^2 \sim \frac{\sum_{i=0}^{\infty} \sum_{i_0=0}^L (i - i_0)^2 \rho_{ai}}{\sum_{i=0}^{\infty} \sum_{i_0=0}^L \rho_{ai}}, \quad (7)$$

where i_0 is the position of the initial active ($z \geq z_c$) site, t is the number of steps measured in the time scale of the avalanche, and $\rho_{ai} = 1$ ($\rho_{ai} = 0$) in active (inactive) sites. Moreover, from the log-log plot of ξ vs $p_c - p$ one can obtain a numerical estimate of p_c and ν .

$d = 1$: The BTW model ($p = 1$) in one dimension exhibits trivial critical behavior [1]. No power-law behavior is observed in the avalanche size and duration distributions; however, one can compute the exponents D and z from the scaling of the moments with system size, resulting in $D \approx 2$ and $z \approx 1$. Using these values and the scaling relations in Eqs. (3) and (4) one could obtain the exponents τ_s and τ_t ($\tau_s = \tau_t = 1$); however, these scaling laws are not valid in this case because the distributions of avalanche size and duration do not follow the scaling law in Eq. (1). On the contrary, the RSM in one dimension has nontrivial behavior. The q dependence of σ_s and σ_t , for different values of $p < 1$, is shown in Figs. 1 and 2, respectively. The numerical estimates of the scaling exponents are given in Table I. For $p_c < p \leq 0.8$ we observe that $\sigma_s(q)$, D , τ_s , and τ_t are practically insensitive to changes in p , but systematic deviations are observed for $\sigma_t(q)$ and z . For $p = 0.9$ the scaling exponents are between those for $p = 1$ and $p = 0.8$. On the other hand, using the

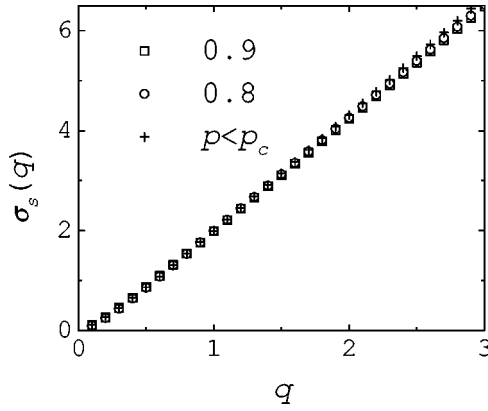


FIG. 1. Plot of $\sigma_s(q)$ in $d=1$ for different values of p . Data above $p_c=0.707(2)$ were obtained using lattice sizes $L=80, 160, 320,$ and 640 . Data below p_c were obtained using probabilities $p=0.670, 0.688, 0.696,$ and 0.7000 .

numerical data below p_c , we have obtained $p_c=0.707 \pm 0.002$, $\nu=1.07 \pm 0.03$, and $z=1.57 \pm 0.02$ that are, within the numerical error, identical to the series-expansion estimates for site DP in a square lattice [13]. Moreover, the exponents $\delta=\tau_t-1$ and z are consistent with previous numerical simulations [3]. We have also carried out finite-size scaling of the distributions of avalanche size and duration. In all cases, including the BTW limit, we observe a good data collapse and the obtained scaling exponents are in agreement with those obtained from the moment analysis. Moreover, within the numerical error, the scaling relations in Eqs. (3) and (4) are satisfied.

$d=2$: The BTW model has nontrivial exponents. However, the data collapse was not compatible with the scaling assumption in Eq. (1). The corrections to scaling in the BTW has been found to be very strong [7,9], making necessary the use of very large lattice sizes to obtain accurate estimates of the scaling exponents. The largest lattice size used in our simulations, $L=512$, seems to be not large enough. Using lattice sizes ranging from $L=512$ to $L=2048$ Chessa *et al.* [9] have obtained a good finite-size scaling for the distribution of avalanche size, but have not for the distribution of avalanche duration. We thus rule out the possibility of determining the BTW exponents in two dimensions with such small lattice sizes. Instead of that, we are going to use the

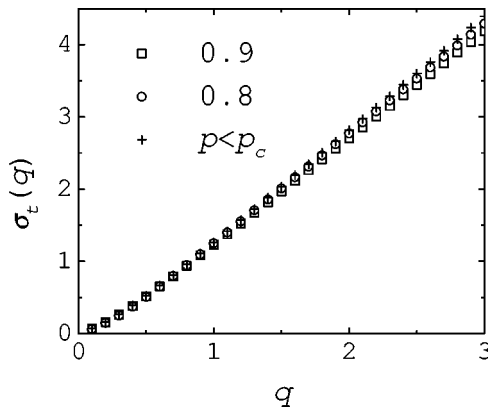


FIG. 2. Plot of $\sigma_t(q)$ in $d=1$. The lattice sites and probabilities used are the same as Fig. 1.

TABLE I. Scaling exponents in $d=1$ for different values of p . Numbers in parentheses are uncertainty figures.

p	D	z	τ_s	τ_t
1	2	1		
0.9	2.25(1)	1.48(2)	1.12(1)	1.17(2)
0.8	2.27(1)	1.52(2)	1.12(1)	1.18(2)
0.708	2.27(1)	1.54(2)	1.13(1)	1.18(2)
$p < p_c$	2.34(1)	1.57(2)	1.16(1)	1.20(2)

numerical estimates by Chessa *et al.* [9] and Lübeck and Usadel [7]. On the contrary, the RSM displays good finite-size scaling for the lattice sizes we have used. Moreover, the scaling exponents obtained from the finite size scaling are in agreement with those obtained from the moment analysis. The q dependence of $\sigma_s(q)$ and $\sigma_t(q)$ is shown in Figs. 3 and 4, respectively. The numerical estimates of the scaling exponents for different values of $p < 1$ are given in Table II, together with the reports by Chessa *et al.* [9] and Lübeck and Usadel [7] for the BTW model. $\sigma_s(q)$, D , τ_s , and τ_t are practically insensitive to changes in p above p_c , even considering the deterministic limit. On the contrary, $\sigma_t(q)$ and z suffer from systematic deviations with changing p . In this case we must be more careful because the finite-size effects have stronger influence on the distribution of avalanche duration. For instance, for the largest lattice size used $L=512$ the distribution of avalanche sizes covers about six decades while the distribution of avalanche durations covers less than five decades. To obtain more precise determination of the dynamic scaling exponents we must increase system size. In the mean time, the scaling exponents of the distribution of avalanche sizes indicate that the RSM in the range $p_c < p < 1$ belongs to the same universality class of the BTW model, which correspond with the limit $p=1$. On the other hand, below p_c we have obtained $p_c=0.344 \pm 0.001$, $\nu=0.728 \pm 0.002$, and $z=1.76 \pm 0.02$ which are, within the numerical error, identical to the numerical estimates for site DP in a body-centered-cubic (bcc) lattice [14]. In this case the difference between the exponents below and above p_c is significant, showing that the RSM above and below p_c belongs to different universality classes.

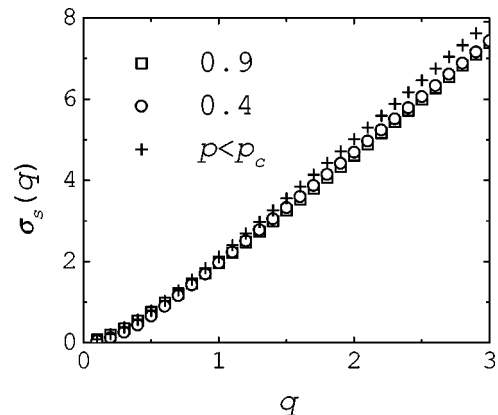


FIG. 3. Plot of $\sigma_s(q)$ in $d=2$ for different values of p . Data above $p_c=0.344(1)$ where obtained using lattice sizes $L=64, 128, 256,$ and 512 . Data below p_c where obtained using probabilities $p=0.26, 0.30, 0.33,$ and 0.34 .

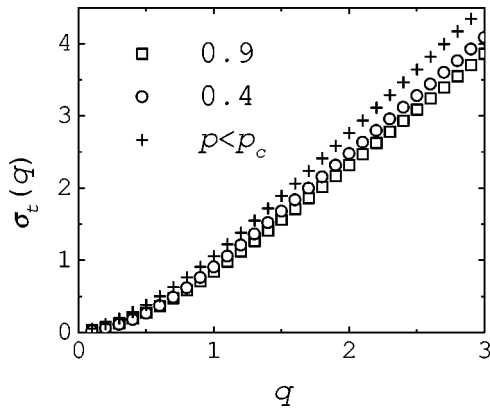


FIG. 4. Plot of $\sigma_\tau(q)$ in $d=2$. The lattice sites and probabilities used are the same as Fig. 3.

The numerical simulations in $d=2$ corroborate that the RSM is similar to DP below p_c . As in $d=1$, the exponents ν and z and the critical probability p_c are identical to the DP values in the corresponding lattice. On the other hand, the difference between the exponents D and z , obtained using the data below p_c and those obtained above but close to p_c , is far from being contained within the error bars. The model above and below p_c belong to different universality classes. Below p_c it is DP with open boundaries and random initial seed, while above p_c we will continue using the term RSM.

The numerical evidence obtained for the avalanche size distribution indicates that in $d=2$ the RSM belongs to the universality class of the BTW model. The scaling exponents τ_s , τ_t , and D are practically independent of p in the SOC

TABLE II. Scaling exponents in $d=2$ for different values of p . Numbers in parentheses are uncertainty figures.

p	D	z	τ_s	τ_t
1	2.73(2) [9]	1.52(2) [9]	1.293 [7]	1.480 [7]
0.9	2.74(1)	1.53(2)	1.28(1)	1.48(2)
0.8	2.75(1)	1.54(2)	1.29(1)	1.48(2)
0.6	2.74(1)	1.58 ^a	1.29(1)	1.47 ^a
0.4	2.74(1)	1.61 ^a	1.28(1)	1.46 ^a
$p < p_c$	2.90(1)	1.76(2)	1.27(1)	1.43(2)

^aThese exponents may be affected by finite-size corrections.

regime $p_c < p \leq 1$; however, z shows a strong p dependence, which may be attributed to finite-size effects. On the contrary in $d=1$ the distribution of avalanche sizes and duration for the BTW model display trivial behavior while in the RSM they satisfy the scaling hypothesis. In larger dimensions $d > 2$ we expect that the SOC regime of the RSM belongs to the BTW universality class as in $d=2$.

In summary the RSM has three different regimes. (i) $0 < p < p_c$ similar to DP, (ii) $p_c < p < 1$ where the toppling rule are still stochastic but the system is in a SOC state, and (iii) $p=1$ (BTW) the toppling rules are deterministic. Based on the moment analysis of the distribution of avalanche sizes we conclude that for $0 < p < p_c$ the model belongs to the DP universality class while for $p_c < p < 1$ it belongs to the BTW universality class.

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